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The state vector methods for space axisymmetric problems in multilayered piezoelectric media

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Abstract

The state vector equations for space axisymmetric problems of transversely isotropic piezoelectric media are established from the basic equations. Using the Hankel transform, the state vector equations are reduced to a system of ordinary differential equations. An analytical solution of the problems in the Hankel transform space is presented in the form of the product of initial state vector and transfer matrix. The transfer matrices are given for the three distinct eigenvalues. Applications of the solutions are discussed. An analytical solution for the transversely isotropic semi-infinite piezoelectric media subjected to concerted point loads on the surface $z = 0$ is presented in the Hankel transform space. Using transfer matrix and the continuity conditions at the layer interfaces, the general solution formulation of N-layered transversely isotropic piezoelectric media is given. A selected set of numerical solutions is presented for a layered semi-infinite piezoelectric solid.

Keywords: Multilayered piezoelectric media; Transversely isotropy; Axisymmetric problems; The state vector equations; Transfer matrix; Analytical solutions

1. Introduction

Smart structures have been currently developed a new research field. Because piezoelectric materials exhibit electromechanical coupling phenomenon, they have been widely used in the field of electroacoustics, transducers and control of structure vibration, etc. Applications of piezoelectric materials have greatly increased the development of theoretical research. The study of piezoelectricity has some important progress.

Chen (1993a,b) presented the fundamental solutions of anisotropic piezoelectric media using the three-dimensional Fourier transform. Dunn (1994, 1996) gave an explicit fundamental solution for an infinite transversely isotropic piezoelectric media. Rajapakse (1997) gave the fundamental solution of the plane

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problems for transversely isotropic piezoelectric media. Wang (1994, 1995) presented the general solutions and the solution of half space problems for transversely isotropic piezoelectric media subjected to point loads. Ding (1996, 1997) have systematically studied the general solution and the solution of half space problems subjected to a point force and a point electric charge for transversely isotropic piezoelectric media.

The state space method is an important method in analysis of multilayered structures (Bahar, 1972, 1975; Bufler, 1971; Sosa and Castro, 1993; Chen et al., 1997; Benitez and Rosakis, 1987; Lee and Jiang, 1996; Wang and Fang, 1999). In this paper, the state vector equations have been established for space axisymmetric problems of transversely isotropic piezoelectric media in a system of cylindrical coordinates by introducing the state vector. Using the Hankel integral transform, the state vector equations presented are reduced to a set of the first order ordinary differential equations. Using the matrix method, the analytical solutions for piezoelectric media of single layers are presented in the form of product of the state vector and the transfer matrix. The transfer matrices are given for the three distinct eigenvalues. Applications of the state vector solutions are discussed. An analytical solution for a semi-infinite piezoelectric medium subjected to the vertical point force P_z and point electric charge Q at the origin of the surface $z = 0$ is presented. According to the continuity conditions at the layer interfaces, the general solution formulation for the space axisymmetric problems of N-layered transversely isotropic piezoelectric media is discussed. The computational formulation is presented. A selected set of numerical solutions is presented for a layered semi-infinite piezoelectric solid.

2. Basic equations

In the absence of body forces and free charges, the governing equations of three-dimensional piezoelectricity can be written in compact form as follows:

$$\begin{aligned}
&\sigma_{ij,j} = 0, \qquad D_{i,i} = 0 \\
&\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \qquad D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k \\
&\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad E_i = -\phi_{,i}
\end{aligned} \tag{1}$$

where σ_{ij}, ε_{ij}, D_i, u_i, E_i are the components, respectively, of stress, strain, electrical displacement, mechanical displacement and electric field, ϕ is the electric potential function. C_{ijkl}, e_{ijk}, ϵ_{ij} are the elastic constants, the piezoelectric constants and the dielectric constants, respectively.

A layered piezoelectric medium consists of horizontal layered, homogeneous and transversely isotropic solids. Assume that the z-axis is perpendicular to the isotropic plane. In a system of cylindrical coordinate, the governing equations of space axisymmetric problems are as follows:

(a) The governing field equations

$$\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\
&\frac{\partial \tau_{zr}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} = 0 \\
&\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0
\end{aligned} \tag{2}$$

(b) The constitutive equations

$$
\begin{aligned}
\sigma_r &= C_{11}\frac{\partial u_r}{\partial r} + C_{12}\frac{u_r}{r} + C_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z} \\
\sigma_\theta &= C_{12}\frac{\partial u_r}{\partial r} + C_{11}\frac{u_r}{r} + C_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z} \\
\sigma_z &= C_{13}\frac{\partial u_r}{\partial r} + C_{13}\frac{u_r}{r} + C_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \phi}{\partial z} \\
\tau_{zr} &= C_{44}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + e_{15}\frac{\partial \phi}{\partial r} \\
D_r &= e_{15}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) - \epsilon_{11}\frac{\partial \phi}{\partial r} \\
D_z &= e_{31}\frac{\partial u_r}{\partial r} + e_{31}\frac{u_r}{r} + e_{33}\frac{\partial u_z}{\partial z} - \epsilon_{33}\frac{\partial \phi}{\partial z}
\end{aligned}
\tag{3}
$$

3. The state vector equations and its solutions

In Eqs. (2) and (3) σ_r, σ_θ, σ_z, τ_{zr}, D_r, D_z, u_r, u_z, ϕ are coupled each other. The state vector equations are a set of partial differential equations, which consist of describing physical phenomena in terms of the minimum possible number of variables. Towards this end, using the governing equations in a system of cylindrical coordinates, σ_r, σ_θ and D_r are eliminated. Taking u_r, u_z, σ_z, τ_{zr}, D_z and ϕ as the state variables, we may obtain a set of partial differential equations as follows:

$$
\begin{aligned}
\frac{\partial u_z}{\partial z} &= -\alpha b_1\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u_r + \epsilon_{33} b_1 \sigma_z + e_{33} b_1 D_z \\
\frac{\partial u_r}{\partial z} &= -\frac{\partial u_z}{\partial r} + b_2 \tau_{zr} - e_{15} b_2 \frac{\partial \phi}{\partial r} \\
\frac{\partial \sigma_z}{\partial z} &= -\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau_{zr} \\
\frac{\partial \tau_{zr}}{\partial z} &= -s b_1\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)u_r - \alpha b_1 \frac{\partial \sigma_z}{\partial r} - \beta b_1 \frac{\partial D_z}{\partial r} \\
\frac{\partial \phi}{\partial z} &= -\beta b_1\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)u_r + e_{33} b_1 \sigma_z - C_{33} b_1 D_z \\
\frac{\partial D_z}{\partial z} &= -e_{15} b_1\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau_{zr} + \kappa b_1\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\phi
\end{aligned}
\tag{4}
$$

in which

$$
\begin{aligned}
&\alpha = C_{13}\epsilon_{33} + e_{33}e_{31}, \qquad \beta = C_{13}e_{33} - C_{33}e_{31}, \qquad \delta = C_{33}\epsilon_{33} + e_{33}^2, \qquad b_1 = \frac{1}{\delta}, \\
&\kappa = e_{15}^2 + \epsilon_{11}C_{44}, \qquad s = C_{11}\delta - C_{13}\alpha - e_{31}\beta, \qquad b_2 = \frac{1}{C_{44}}
\end{aligned}
\tag{5}
$$

Eq. (4) is a system of partial differential equations taking the state variables u_z, u_r, σ_z, τ_{rz}, D_z and ϕ as the basic unknowns. Using the Hankel integral transform, Eq. (4) can be reduced to a set of ordinary differential equations. Using the following Hankel transform, we have

$$\begin{aligned}\langle \tilde{u}_z \quad \tilde{\sigma}_z \quad \tilde{\phi} \quad \widetilde{D}_z \rangle &= \int_0^\infty \langle u_z \quad \sigma_z \quad \phi \quad D_z \rangle r J_0(\xi r)\,\mathrm{d}r \\ \langle \tilde{u}_r \quad \tilde{\tau}_{zr} \rangle &= \int_0^\infty \langle u_r \quad \tau_{zr} \rangle r J_1(\xi r)\,\mathrm{d}r\end{aligned} \tag{6}$$

The inversion of the Hankel transforms is as follows:

$$\begin{aligned}\langle u_z \quad \sigma_z \quad \phi \quad D_z \rangle &= \int_0^\infty \langle \tilde{u}_z \quad \tilde{\sigma}_z \quad \tilde{\phi} \quad \widetilde{D}_z \rangle \xi J_0(\xi r)\,\mathrm{d}\xi \\ \langle u_r \quad \tau_{zr} \rangle &= \int_0^\infty \langle \tilde{u}_r \quad \tilde{\tau}_{zr} \rangle \xi J_1(\xi r)\,\mathrm{d}\xi\end{aligned} \tag{7}$$

Substituting Eq. (6) into Eq. (4) and using the derivative relations of the Bessel function, we obtain a set of ordinary differential equations in the Hankel transform space.

$$\frac{\mathrm{d}\tilde{\eta}(\xi, z)}{\mathrm{d}z} = A(\xi)\tilde{\eta}(\xi, z) \tag{8}$$

in which

$$\tilde{\eta}(\xi, z) = [\tilde{u}_z \quad \tilde{u}_r \quad \tilde{\sigma}_z \quad \tilde{\tau}_{zr} \quad \tilde{\phi} \quad \widetilde{D}_z]^{\mathrm{T}}$$

$$A(\xi) = \begin{bmatrix} 0 & -b_1\alpha\xi & b_1\epsilon_{33} & 0 & 0 & b_1 e_{33} \\ \xi & 0 & 0 & b_2 & b_2 e_{15}\xi & 0 \\ 0 & 0 & 0 & -\xi & 0 & 0 \\ 0 & b_1 s\xi^2 & b_1\alpha\xi & 0 & 0 & b_1\beta\xi \\ 0 & -b_1\beta\xi & b_1 e_{33} & 0 & 0 & -b_1 C_{33} \\ 0 & 0 & 0 & -b_2 e_{15}\xi & -b_2\kappa\xi^2 & 0 \end{bmatrix} \tag{9}$$

According to the theory of ordinary differential equations, the solution of Eq. (8) can be expressed in the following form.

$$\tilde{\eta}(\xi, z) = \exp(zA(\xi))\tilde{\eta}(\xi, 0) \tag{10}$$

where $\exp(zA(\xi))$ is the transfer matrix between the initial state vector $\tilde{\eta}(\xi, 0)$ and the state vector $\tilde{\eta}(\xi, z)$ of arbitrary depth z. According to the Cayley–Hamilton theorem (Bellman, 1970), we have

$$\exp(zA(\xi)) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + a_4 A^4 + a_5 A^5 \tag{11}$$

in which a_0, a_1, a_2, a_3, a_4, a_5 are related to the eigenvalues λ of the matrix $A(\xi)$. The characteristic equations of the matrix $A(\xi)$ can be expressed as follows:

$$a\lambda^6 - b\lambda^4\xi^2 + c\lambda^2\xi^4 - d\xi^6 = 0 \tag{12}$$

in which

$$\begin{aligned} a &= C_{44}(C_{33}\epsilon_{33} + e_{33}^2) \\ b &= C_{33}[\epsilon_{11}C_{44} + C_{11}\epsilon_{33} + e_{31}(e_{31} + e_{15})] + (e_{31} + e_{15})(C_{33}e_{15} - 2C_{13}e_{33}) \\ &\quad + e_{33}(C_{11}e_{33} - 2C_{44}e_{31}) - C_{13}\epsilon_{33}(C_{13} + 2C_{44}) \\ c &= C_{11}(\epsilon_{11}C_{33} + 2e_{15}e_{33}) + C_{44}(C_{11}\epsilon_{33} + e_{31}^2) - \epsilon_{11}C_{13}(C_{13} + 2C_{44}) - 2e_{15}C_{13}(e_{31} + e_{15}) \\ d &= C_{11}(e_{15}^2 + \epsilon_{11}C_{44}) \end{aligned} \tag{13}$$

The roots of Eq. (12) can be obtained in the analytical form. Assume that the characteristic roots of the matrix $A(\xi)$ are equal to λ_j $(j = 1, 2, 3)$, $\lambda_4 = -\lambda_1$, $\lambda_5 = -\lambda_2$, $\lambda_6 = -\lambda_3$ respectively. λ_j are related to the elastic constants, the piezoelectric constants and the dielectric constants. The charateristic roots of Eq.(12) satisfy $\mathrm{Re}[\lambda_j] > 0$. a_0, a_1, a_2, a_3, a_4, a_5 for the three cases of the characteristic roots are expressed as follows:

(i) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$

$$
\begin{aligned}
& a_0 = \sum_{k=1}^{3} h_k \lambda_{k+1}^2 \lambda_{k+2}^2, \qquad a_1 = \sum_{k=1}^{3} g_k \lambda_{k+1}^2 \lambda_{k+2}^2, \qquad a_2 = -\sum_{k=1}^{3} h_k (\lambda_{k+1}^2 + \lambda_{k+2}^2), \\
& a_3 = -\sum_{k=1}^{3} g_k (\lambda_{k+1}^2 + \lambda_{k+2}^2), \qquad a_4 = \sum_{k=1}^{3} h_k, \qquad a_5 = \sum_{k=1}^{3} g_k, \qquad h_k = \frac{ch\lambda_k z}{d_k}, \\
& g_k = \frac{sh\lambda_k z}{\lambda_k d_k}, \qquad d_k = (\lambda_{k+1}^2 - \lambda_k^2)(\lambda_{k+2}^2 - \lambda_k^2)
\end{aligned} \tag{14}
$$

(ii) $\lambda_1 = \lambda_2 = \lambda_3$

$$
\begin{aligned}
& a_0 = \frac{1}{8}(8 + \lambda_1^2 z^2) ch\lambda_1 z - \frac{5}{8}\lambda_1 z sh\lambda_1 z, \qquad a_1 = \frac{1}{8\lambda_1}[(15 + \lambda_1^2 z^2) sh\lambda_1 z - 7\lambda_1 z ch\lambda_1 z], \\
& a_2 = \frac{1}{4\lambda_1}(3z sh\lambda_1 z - \lambda_1 z^2 ch\lambda_1 z), \qquad a_3 = \frac{1}{4\lambda_1^3}[5\lambda_1 z ch\lambda_1 z - (5 + \lambda_1^2 z^2) sh\lambda_1 z], \\
& a_4 = \frac{1}{8\lambda_1^3}(\lambda_1 z^2 ch\lambda_1 z - z sh\lambda_1 z), \qquad a_5 = \frac{1}{8\lambda_1^5}[(3 + \lambda_1^2 z^2) sh\lambda_1 z - 3\lambda_1 z ch\lambda_1 z]
\end{aligned} \tag{15}
$$

(iii) $\lambda_1 \neq \lambda_2 = \lambda_3 \neq \lambda_1$

$$
\begin{aligned}
& a_0 = \frac{1}{2\lambda_2(\lambda_1^2 - \lambda_2^2)^2}[-\lambda_1^2\lambda_2^2(\lambda_1^2 - \lambda_2^2) z sh\lambda_2 z + 2\lambda_2^5 ch\lambda_1 z + 2\lambda_1^2\lambda_2(\lambda_1^2 - 2\lambda_2^2) ch\lambda_2 z] \\
& a_1 = \frac{1}{2\lambda_1\lambda_2^4(\lambda_1^2 - \lambda_2^2)^2}[2\lambda_2^8 sh\lambda_1 z + \lambda_1\lambda_2^3(3\lambda_1^4 - 5\lambda_1^2\lambda_2^2) sh\lambda_2 z - \lambda_1^3\lambda_2^4(\lambda_1^2 - \lambda_2^2) z ch\lambda_2 z] \\
& a_2 = \frac{1}{2\lambda_2(\lambda_1^2 - \lambda_2^2)^2}[(\lambda_1^4 - \lambda_2^4) z sh\lambda_2 z - 4\lambda_2^3(ch\lambda_1 z - ch\lambda_2 z)] \\
& a_3 = \frac{1}{2\lambda_1\lambda_2^4(\lambda_1^2 - \lambda_2^2)^2}[-4\lambda_2^6 sh\lambda_1 z - \lambda_1\lambda_2(\lambda_1^4 - 5\lambda_2^4) sh\lambda_2 z + \lambda_2^2\lambda_1(\lambda_1^4 - \lambda_2^4) z ch\lambda_2 z] \\
& a_4 = \frac{1}{2\lambda_2(\lambda_1^2 - \lambda_2^2)^2}[2\lambda_2(ch\lambda_1 z - ch\lambda_2 z) - (\lambda_1^2 - \lambda_2^2) z sh\lambda_2 z] \\
& a_5 = \frac{1}{2\lambda_1\lambda_2^4(\lambda_1^2 - \lambda_2^2)^2}[2\lambda_2^4 sh\lambda_1 z + \lambda_1\lambda_2(\lambda_1^2 - 3\lambda_2^2) sh\lambda_2 z - \lambda_2^2\lambda_1(\lambda_1^2 - \lambda_2^2) z ch\lambda_2 z]
\end{aligned} \tag{16}
$$

Using Eqs. (9), (11), (15) and (16), the transfer matrix $[G(\xi, z)]$ from the initial state vector $\tilde{\eta}(\xi, 0)$ to the state vector $\tilde{\eta}(\xi, z)$ of any depth can obtained.

$$
\tilde{\eta}(\xi, z) = [G(\xi, z)]\tilde{\eta}(\xi, 0) \tag{17}
$$

in which $[G(\xi, z)]$ is 6×6 matrix.

$$
\begin{aligned}
&G_{11} = a_0 - b_1\alpha\xi^2 a_2 + b_1^2 b_2 \xi^4 \beta_1 a_4, \qquad G_{12} = -b_1\alpha a_1 \xi + b_1^2 b_2 \beta_1 \xi^3 a_3 + b_1^3 b_2^2 \theta_1 \xi^5 a_5,\\
&G_{21} = -G_{34}, \qquad G_{22} = a_0 + b_1 b_2 \alpha_3 \xi^2 a_2 + b_1^2 b_2^2 \gamma_3 \xi^4 a_4, \qquad G_{31} = -b_1 s \xi^4 a_3 - b_1^2 b_2 \beta_6 \xi^6 a_5,\\
&G_{32} = -b_1 s \xi^3 a_2 - b_1^2 b_2 \beta_6 \xi^5 a_4, \qquad G_{41} = -G_{32}, \qquad G_{42} = b_1 s \xi^2 a_1 + b_1^2 b_2 \beta_6 \xi^4 a_3 + b_1^3 b_2^2 \theta_6 \xi^6 a_5,\\
&G_{51} = -b_1 \beta \xi^2 a_2 - b_1^2 b_2 \beta_7 \xi^4 a_4, \qquad G_{52} = -b_1 \beta \xi a_1 - b_1^2 b_2 \beta_7 \xi^3 a_3 - b_1^3 b_2^2 \theta_7 \xi^5 a_5,\\
&G_{61} = -b_1 b_2 \alpha_5 \xi^4 a_3 - b_1^2 b_2^2 \gamma_5 \xi^6 a_5, \qquad G_{62} = -b_1 b_2 \alpha_5 \xi^3 a_2 - b_1^2 b_2^2 \gamma_5 \xi^5 a_4,\\
&G_{13} = b_1 \epsilon_{33} a_1 - b_1^2 b_2 \beta_2 \xi^2 a_3 + b_1^3 b_2^2 \theta_2 \xi^4 a_5, \qquad G_{14} = -b_1 b_2 \alpha_1 \xi a_2 + b_1^2 b_2^2 \gamma_1 \xi^3 a_4,\\
&G_{23} = -G_{14}, \qquad G_{24} = b_2 a_1 + b_1 b_2 \beta_4 \xi^2 a_3 + b_1^2 b_2^3 \theta_4 \xi^4 a_5, \qquad G_{33} = G_{11},\\
&G_{34} = -\xi a_1 - b_1 b_2 \alpha_3 \xi^3 a_3 - b_1^2 b_2^2 \gamma_3 \xi^5 a_5, \qquad G_{43} = -G_{12}, \qquad G_{44} = G_{22},\\
&G_{53} = b_1 e_{33} a_1 + b_1^2 b_2 \beta_3 \xi^2 a_3 + b_1^3 b_2^2 \theta_3 \xi^4 a_5, \qquad G_{54} = -b_1 b_2 \alpha_4 \xi a_2 - b_1^2 b_2^2 \gamma_4 \xi^3 a_4,\\
&G_{63} = -b_1 b_2 \alpha_2 \xi^2 a_2 + b_1^2 b_2^2 \gamma_2 \xi^4 a_4, \qquad G_{64} = -b_2 e_{15} \xi a_1 - b_1 b_2^2 \beta_5 \xi^3 a_3 - b_1^2 b_2^3 \theta_5 \xi^5 a_5,\\
&G_{15} = G_{63}, \qquad G_{16} = G_{53}, \qquad G_{25} = -G_{64}, \qquad G_{26} = -G_{54}, \qquad G_{35} = G_{61}, \qquad G_{36} = G_{51},\\
&G_{45} = -G_{62}, \qquad G_{46} = -G_{52}, \qquad G_{55} = G_{66}, \qquad G_{55} = a_0 + b_1 b_2 \alpha_6 \xi^2 a_2 + b_1^2 b_2^2 \gamma_6 \xi^4 a_4,\\
&G_{56} = -b_1 C_{33} a_1 - b_1^2 b_2 \beta_8 \xi^2 a_3 - b_1^3 b_2^2 \theta_8 \xi^4 a_5, \qquad G_{65} = -b_2 \kappa \xi^2 a_1 - b_1 b_2^2 \beta_9 \xi^4 a_3 - b_1^2 b_2^3 \theta_9 \xi^6 a_5,
\end{aligned} \tag{18}
$$

in which α_i, β_i, γ_i, θ_i are referred to the Appendix A.

4. Applications of the state vector solutions

4.1. Semi-infinite piezoelectric solid

Eq. (17) represents the transfer relation between the initial state vector and the state vector of any depth. Using Eq. (17), the solution of space axisymmetric problems of transversely isotropic piezoelectric media can be obtained. In the following, semi-infinite piezoelectric solids subjected to axisymmetric point load and point charge is considered. In this problem, there is

$$\lim_{z\to\infty}[\sigma_z \quad \sigma_r \quad \sigma_\theta \quad \tau_{zr} \quad D_z \quad D_r \quad u_z \quad u_r \quad \phi] \to 0 \tag{19}$$

In Eq. (17), each element of the transfer matrix $[G(\xi,z)]$ included $e^{-\lambda_j z}$ and $e^{\lambda_j z}$. It is quite evident that $e^{\lambda_j z}$ is not coincided with Eq. (19). So the terms included $e^{\lambda_j z}$ for each element in $[G(\xi,z)]$ must be deleted. The modified transfer matrix $[G^*(\xi,z)]$ is obtained. The transfer relation between the initial state vector and the state vector of any depth for the semi-infinite piezoelectric media is as follows:

$$\tilde{\eta}(\xi,z) = [G^*(\xi,z)]\tilde{\eta}(\xi,0) \tag{20}$$

For the semi-infinite piezoelectric media subjected to the mechanical and electric loads on the surface, the state vector at z can be not obtained only using Eq. (20) because the state variables $\tilde{u}_z(\xi,0)$, $\tilde{u}_r(\xi,0)$ and $\tilde{\phi}(\xi,0)$ are unknown. Using the boundary conditions of the surface $z=0$, the relations between the generalized displacements and generalized stresses can be obtained. According to Eq. (20), we have

$$\begin{Bmatrix} \tilde{u}_z(\xi,0) \\ \tilde{u}_r(\xi,0) \\ \tilde{\phi}(\xi,0) \end{Bmatrix} = \begin{bmatrix} G^*_{11} & G^*_{12} & G^*_{15} & G^*_{13} & G^*_{14} & G^*_{16} \\ G^*_{21} & G^*_{22} & G^*_{25} & G^*_{23} & G^*_{24} & G^*_{26} \\ G^*_{51} & G^*_{52} & G^*_{55} & G^*_{53} & G^*_{54} & G^*_{56} \end{bmatrix}_{z=0} \begin{Bmatrix} \tilde{u}_z(\xi,0) \\ \tilde{u}_r(\xi,0) \\ \tilde{\phi}(\xi,0) \\ \tilde{\sigma}_z(\xi,0) \\ \tilde{\tau}_{zr}(\xi,0) \\ \widetilde{D}_z(\xi,0) \end{Bmatrix} \tag{21}$$

Solving Eq. (21), the following relation can be obtained.

$$\begin{Bmatrix} \tilde{u}_z(\xi,0) \\ \tilde{u}_r(\xi,0) \\ \tilde{\phi}(\xi,0) \end{Bmatrix} = \frac{1}{\xi} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \tilde{\sigma}_z(\xi,0) \\ \tilde{\tau}_{zr}(\xi,0) \\ \widetilde{D}_z(\xi,0) \end{Bmatrix} \tag{22}$$

in which

$$\frac{1}{\xi} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1-G^*_{11} & -G^*_{12} & -G^*_{15} \\ -G^*_{21} & 1-G^*_{22} & -G^*_{25} \\ -G^*_{51} & -G^*_{52} & 1-G^*_{55} \end{bmatrix}^{-1}_{z=0} \begin{bmatrix} G^*_{13} & G^*_{14} & G^*_{16} \\ G^*_{23} & G^*_{24} & G^*_{26} \\ G^*_{53} & G^*_{54} & G^*_{56} \end{bmatrix}_{z=0} \tag{23}$$

The following equation can be obtained from Eq. (22).

$$\tilde{\eta}(\xi,0) = \frac{1}{\xi}[A^*]\tilde{f}(\xi,0) \tag{24}$$

in which

$$[A^*] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \xi & 0 & 0 \\ 0 & \xi & 0 \\ a_{31} & a_{32} & a_{33} \\ 0 & 0 & \xi \end{bmatrix} \qquad \tilde{f}(\xi,0) = \begin{Bmatrix} \tilde{\sigma}_z(\xi,0) \\ \tilde{\tau}_{zr}(\xi,0) \\ \widetilde{D}_z(\xi,0) \end{Bmatrix} = \begin{Bmatrix} -\tilde{p}(\xi) \\ -\tilde{g}(\xi) \\ \tilde{\varpi}(\xi) \end{Bmatrix} \tag{25}$$

In Eq. (25), $\tilde{p}(\xi)$ and $\tilde{g}(\xi)$ are the expression of vertical and horizontal loads acting on the surface $z = 0$ in the Hankel transform space $\tilde{\varpi}(\xi)$ is the expression of charge acting on the surface $z = 0$ in the Hankel transform space. Substituting Eq. (24) into (20), we obtain

$$\tilde{\eta}(\xi,z) = \frac{1}{\xi}[G^*(\xi,z)][A^*]\tilde{f}(\xi,0) \tag{26}$$

Eq. (26) gives the state vector solutions of the Hankel transform space for the semi-infinite piezoelectric media subjected to axisymmetric loads and electric charge. Assume that the vertical point P and point electric charge Q act on the surface $z = 0$, we have

$$\tilde{\eta}(\xi,z) = \frac{1}{\xi}[G^*(\xi,z)][A^*] \begin{Bmatrix} -\frac{P}{2\pi} \\ 0 \\ \frac{Q}{2\pi} \end{Bmatrix} \tag{27}$$

Substituting Eq. (27) into (7), the solution of the physical space for the semi-infinite piezoelectric media subjected to the vertical point force P and the point electric charge Q on the surface $z = 0$ can be obtained. For brevity, the explicit formulas are omitted in this paper.

4.2. Layered piezoelectric solids

For N ($N \geqslant 2$) layered piezoelectric medium, continuous using Eq. (17) and considering continuity conditions at the interfaces, the state vector at the interfaces can be eliminated. Assume that h_i represents the thickness of the ith layer, we have

$$\tilde{\eta}(\xi, h_N) = [G_N(\xi, h_N)][G_{N-1}(\xi, h_{N-1})]\cdots[G_1(\xi, h_1)]\tilde{\eta}(\xi, 0) = [T]\tilde{\eta}(\xi, 0) \tag{28}$$

where $[T] = [G_N(\xi, h_N)][G_{N-1}(\xi, h_{N-1})]\cdots[G_1(\xi, h_1)]$ indicate the transfer matrices of each layer, respectively. Eq. (28) describes the transfer relations between the state vector of the top surface and of the bottom surface.

Using Eq. (28) and introducing boundary conditions, the state vector at $z = 0$ and H can be obtained. Assume that the stresses and the electric displacements at $z = 0$ and H are known. Rearranging Eq. (17), we have

$$\begin{Bmatrix} \tilde{u}(\xi, h_N) \\ \tilde{\sigma}(\xi, h_N) \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \tilde{u}(\xi, 0) \\ \tilde{\sigma}(\xi, 0) \end{Bmatrix} \tag{29}$$

where T_{ij} is 3×3 submatrices. $\tilde{u}$ is the component of displacements and electric potential. $\tilde{\sigma}$ is the component of stress and electric displacement. Solving Eq. (29), the components of displacements and electric potential on the surface $z = 0$ can be expressed in the following form:

$$\tilde{u}(\xi, 0) = -T_{21}^{-1}T_{22}\tilde{\sigma}(\xi, 0) + T_{21}^{-1}\tilde{\sigma}(\xi, h_N) \tag{30}$$

4.3. Layered semi-infinite piezoelectric solids

For N layered semi-infinite piezoelectric solid (Fig. 1), substituting Eq. (24) into (28), we obtain

$$\tilde{\eta}(\xi, h_N) = \frac{1}{\xi}[T][A^*]\tilde{f}(\xi, 0) = \frac{1}{\xi}[T^*]\tilde{f}(\xi, 0) \tag{31}$$

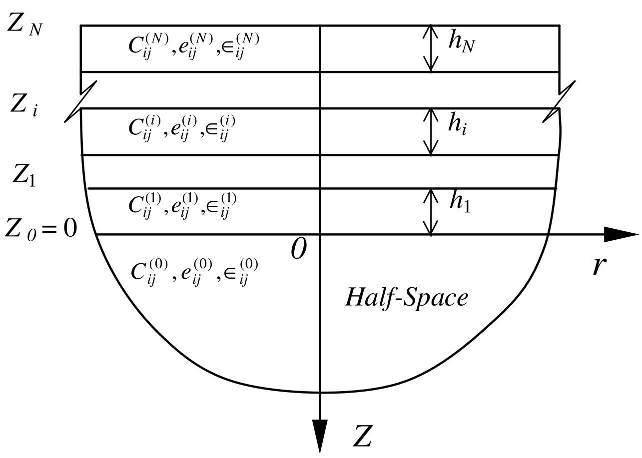

Fig. 1. Layered half-space.

in which $[T^*] = [T][A^*]$ is a 6×3 matrix. The following relation is obtained from Eq. (31):

$$\left\{ \begin{array}{c} \tilde{\sigma}_z(\xi, h_N) \\ \tilde{\tau}_{zr}(\xi, h_N) \\ \widetilde{D}_z(\xi, h_N) \end{array} \right\} = \frac{1}{\xi} \begin{bmatrix} T^*_{31} & T^*_{32} & T^*_{33} \\ T^*_{41} & T^*_{42} & T^*_{43} \\ T^*_{61} & T^*_{62} & T^*_{63} \end{bmatrix} \left\{ \begin{array}{c} \tilde{\sigma}_z(\xi, 0) \\ \tilde{\tau}_{zr}(\xi, 0) \\ \widetilde{D}_z(\xi, 0) \end{array} \right\} \tag{32}$$

Solving the above equation, we have

$$\left\{ \begin{array}{c} \tilde{\sigma}_z(\xi, 0) \\ \tilde{\tau}_{zr}(\xi, 0) \\ \widetilde{D}_z(\xi, 0) \end{array} \right\} = \xi \begin{bmatrix} T^*_{31} & T^*_{32} & T^*_{33} \\ T^*_{41} & T^*_{42} & T^*_{43} \\ T^*_{61} & T^*_{62} & T^*_{63} \end{bmatrix}^{-1} \left\{ \begin{array}{c} \tilde{\sigma}_z(\xi, h_N) \\ \tilde{\tau}_{zr}(\xi, h_N) \\ \widetilde{D}_z(\xi, h_N) \end{array} \right\} \tag{33}$$

Assume $\tilde{\sigma}_z(\xi, h_N)$, $\tilde{\tau}_{zr}(\xi, h_N)$ and $\widetilde{D}_z(\xi, h_N)$ on the top surface are known, the state vector on $z = 0$ can be obtained from Eqs. (33) and (24). Because the state vector at $z = 0$ is known, the state vector of the interfaces for N-layered piezoelectric media can be obtained using Eq. (17). Using Eqs. (17), (3) and (7) and constitutive equations, the stresses, displacements, electric displacements and electric potential function in the field can be computed. The integral of the Bessel function can be computed using the subdomain Gauss quadrature for the Hankel inversion transform.

5. Numerical example

Based on the above discussion and solution expression, computer program had been developed to calculate the response of the layered pizoelectric solid. The subdomain Gauss quadrature is used in the evaluation of semi-infinite integrals concerning the inverse Hankel integral transform. A three-layered ($N = 2$) piezoelectric semi-infinite solid subjected to a circular uniform vertical load or charge is considered. The properties of piezoelectric materials are given in Table 1. Two layered piezoelectric semi-infinite solid with stacking sequences PZT-6B/PZT-4/PZT-6B (called P6/P4/P6) and PZT-4/PZ-6B/PZT-4 (called P4/P6/P4) are investigated. The surface circular uniform load $p = 1$ N/m^2 and the surface circular uniform charge $Q = 1$ C/m are assumed. Radius $a = 1$ m, $h_1 = 0.2$ m, $h_2 = 0.1$ m.

Figs. 2 and 3 show, respectively, the variation of the electric displacement D_z and the normal stress σ_z along the z-axis due to a surface circular uniform charge on the top surface. Figs. 4 and 5 show, respectively, the variation of the electric displacement D_z and the normal stress σ_z along the z-axis due to a surface circular uniform load on the top surface.

The following general features are observed from Figs. 2–5. The normal stress has been greatly influenced by the stacking sequences and the electric displacement relatively small differences for the same circular uniform charge. The electric displacement has been greatly influenced by the stacking sequences and the normal stress relatively small differences for the same circular uniform load. The decay of the vertical stress and electric field with the depth is very rapid in the case of a vertical load and electric charge.

Table 1
Coefficients of the piezoelectric materials (C_{ij} in 10^{10} N/m^2 e_{ij} in C/m^2, ϵ_{ij} in 10^{-9} F/m)

	C_{11}	C_{33}	C_{12}	C_{13}	C_{44}	e_{15}	e_{31}	e_{33}	ϵ_{11}	ϵ_{33}
PZT-4	13.9	11.5	7.78	7.43	2.56	12.7	−5.2	15.1	6.45	5.62
PZT-6B	16.8	16.3	6.0	6.0	2.71	4.6	−0.9	7.1	3.6	3.4

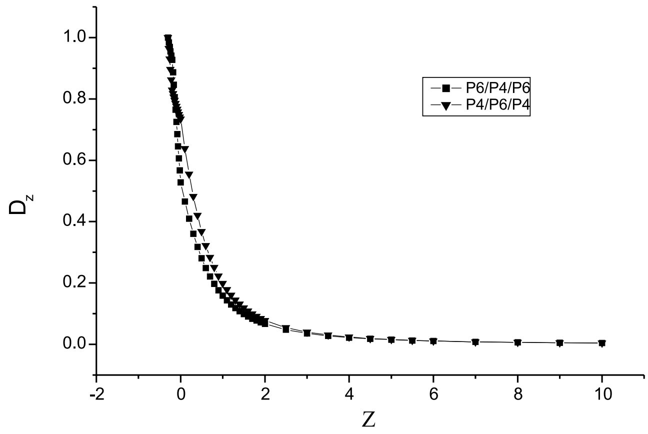

Fig. 2. Variation of the electric displacement D_z along the z-axis subjected to a surface charge on the top surface.

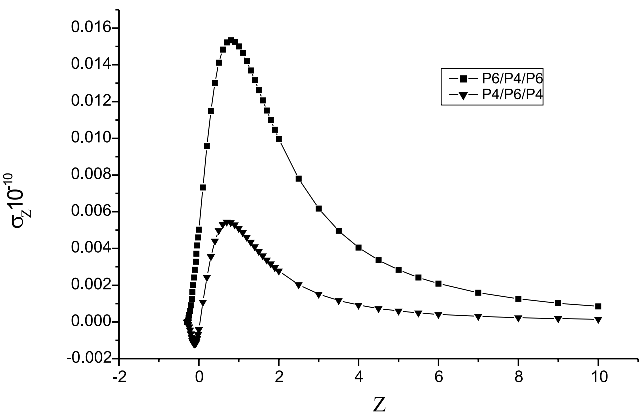

Fig. 3. Variation of the normal stress σ_z along the z-axis subjected to a surface charge on the top surface.

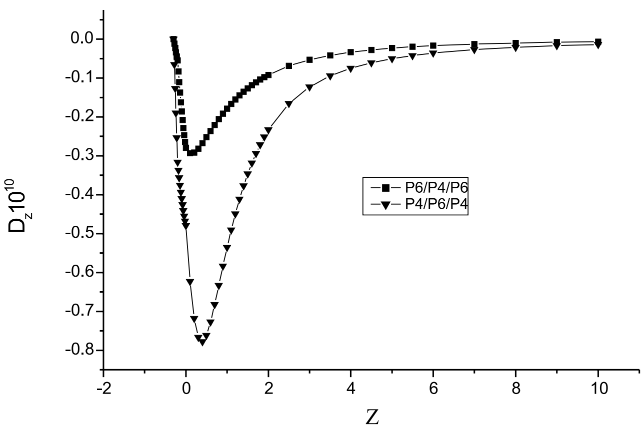

Fig. 4. Variation of the electric displacement D_z along the z-axis subjected to a surface load on the top surface.

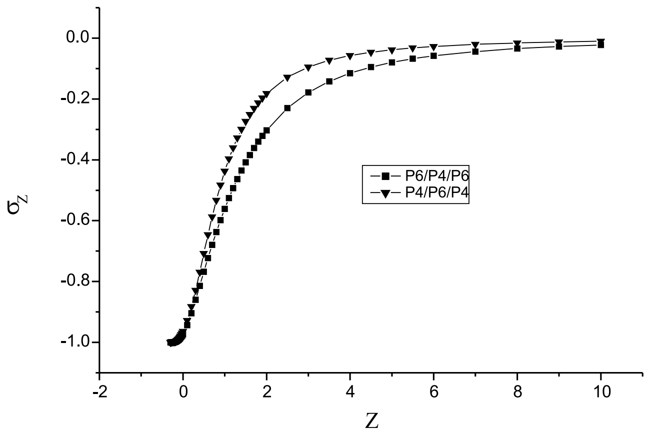

Fig. 5. Variation of the normal stress σ_z along the z-axis subjected to a surface load on the top surface.

6. Conclusions

The state vector formulation for the axisymmetric piezoelectric solid is presented. A complete analytical general solution for the multilayered axisymmetric piezoelectric solid is obtained in a compact form. The derivation and solution presented in this paper are succinct and clear. These general solutions can be used to derive closed form analytical solutions for concentrated point load and an electric charge. The present solutions can be also used in the analysis of inclusions, cracks, defects, etc., related to smart composite structures. As compared with the classical method, the advantage of the present method is that the number of unknowns is not related to the number of layers. The classical method need to solve a $(6N+3)\times(6N+3)$ simultaneous linear equations and the present method only need to solve a 33 simultaneous linear equations at each inversion of the Hankel transform. From the view of computation, the computational efficiency is raised by using the present method. Based on the present solution, the singular solutions of a single and more layers piezoelectric medium can be obtained. Solutions for more complicated load and electric charge distributions can be directly derived from the present solutions through an appropriate substitution or integration.

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Appendix A

$$\alpha_1 = \alpha + \epsilon_{33}C_{44} + e_{33}e_{15}, \qquad \alpha_2 = \alpha e_{15} + e_{33}\kappa, \qquad \alpha_3 = -\alpha C_{44} + s - e_{15}\beta$$

$$\alpha_4 = e_{33}C_{44} + \beta - C_{33}e_{15}, \qquad \alpha_5 = se_{15} - \kappa\beta, \qquad \alpha_6 = \kappa C_{33} - \beta e_{15}$$

$$\beta_1 = \alpha^2 C_{44} - \alpha_1 s + \alpha_2\beta, \qquad \beta_2 = \alpha\epsilon_{33}C_{44} + \alpha_1\alpha + \alpha_2 e_{33}$$

$$\beta_3 = -\alpha e_{33}C_{44} - \alpha_1\beta + \alpha_2 C_{33}, \qquad \beta_4 = \alpha_3 - \alpha_1 C_{44} - \alpha_4 e_{15}$$

$$\beta_5 = \alpha_3 e_{15} - \alpha_4\kappa, \qquad \beta_6 = -\alpha s C_{44} + \alpha_3 s - \alpha_5\beta$$

$$\beta_7 = s e_{33}C_{44} + \alpha_3\beta - C_{33}\alpha_5, \qquad \beta_8 = \beta e_{33}C_{44} + \alpha_4\beta + C_{33}\alpha_6$$

$$\beta_9 = \alpha_5 e_{15} + \kappa\alpha_6, \qquad \gamma_1 = \beta_1 + \beta_2 C_{44} - \beta_3 e_{15}, \qquad \gamma_2 = \beta_1 e_{15} - \beta_3\kappa$$

$$\gamma_3 = -\alpha_3\alpha C_{44} + \beta_4 s - \beta_5\beta, \qquad \gamma_4 = \alpha_3 e_{33}C_{44} + \beta_4\beta - \beta_5 C_{33}$$

$$\gamma_5 = \beta_6 e_{15} - \beta_7\kappa, \qquad \gamma_6 = -e_{15}\beta_7 + \kappa\beta_8$$

$$\theta_1 = -\alpha\beta_1 C_{44} + \gamma_1 s - \gamma_2\beta, \qquad \theta_2 = \beta_1\epsilon_{33} + \gamma_1\alpha + e_{33}\gamma_2$$

$$\theta_3 = \beta_1 e_{33}C_{44} + \gamma_1\beta - \gamma_2 C_{33}, \qquad \theta_4 = \gamma_3 + \gamma_1 C_{44} - e_{15}\gamma_4, \qquad \theta_5 = \gamma_3 e_{15} - \gamma_4\kappa$$

$$\theta_6 = -\alpha\beta_6 C_{44} + s\gamma_3 - \beta\gamma_5, \qquad \theta_7 = e_{33}\beta_6 C_{44} + \beta\gamma_3 - \gamma_5 C_{33}$$

$$\theta_8 = e_{33}\beta_7 C_{44} + \beta\gamma_4 + C_{33}\gamma_6, \qquad \theta_9 = e_{15}\gamma_5 + \kappa\gamma_6$$

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